COMPARTMENT MODEL FOR VALVELESS PUMPING WITH GRAVITY

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Introduction

Motive:* *Valveless pumping and the Liebau phenomenon

- **Valveless pumping**
- **Liebau phenomenon - both positive and negative directions of the net flow (or power) exist due to a simple excitation at an asymmetric position**
- **Physical model with fluid tanks attached to a valveless tube under the gravity**

Energy-Based Lumped Model (ELM):* *a lumped model for valveless pumping based on the energy-power principle

- **Physical law between the energy and power**
- **Physical interpolations for the fluid pressure and flux**
- **Mathematical model describing the underlying physics of fluid**

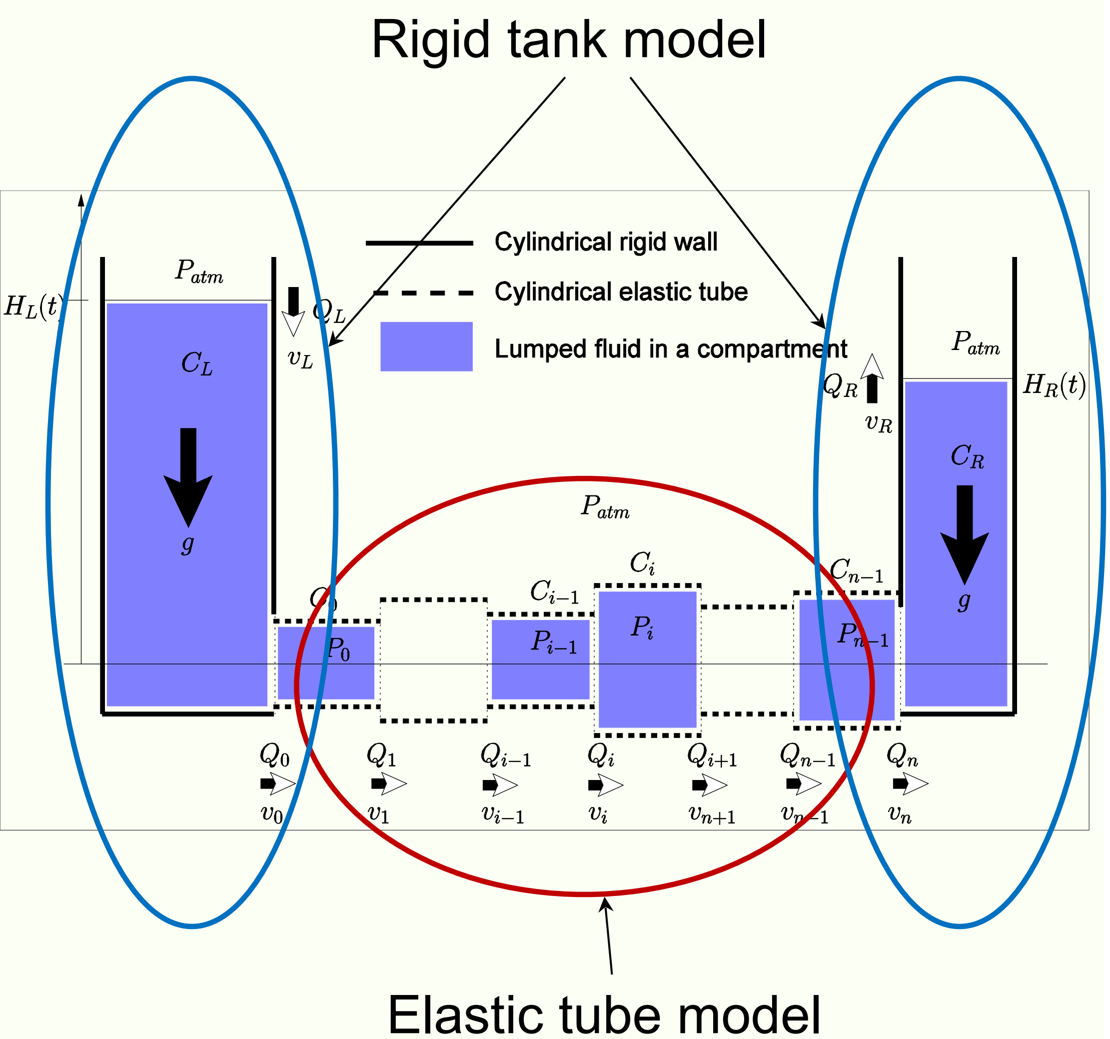
Representative advantages of the ELM

- ***Being able to prove it the energy conservation model***
- ***Being able to analyze the relationship between the energy storing and the net power direction***
- ***It can perfectly produce the apparent phenomena in fluid – reliability of ELM***

Other Models for the valveless pumping

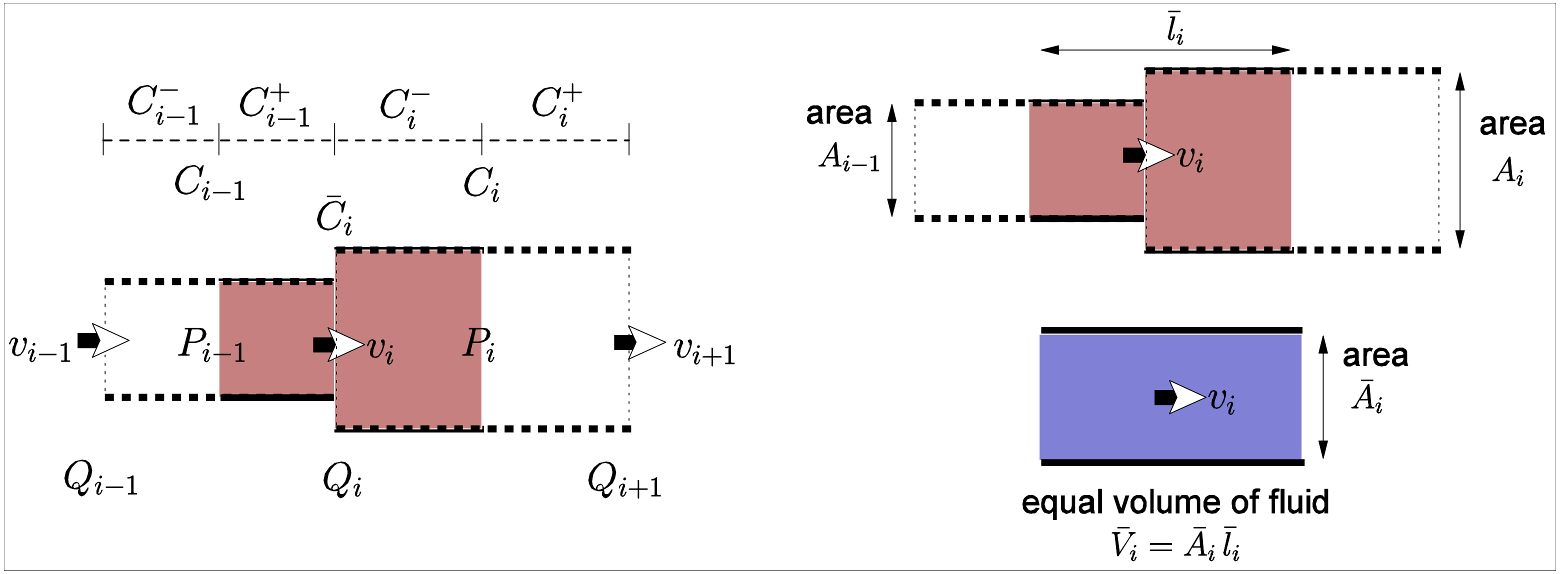
- **C. G. Manopoulos, 2006, Closed loop model, quasi-1D model**
- **T. T. Bringley, 2008, Closed loop model, Experiment**
- **G. Propst, 2003, Liebau phenomena**
- **M. Gharib, 2005, 2008, impedance pump model**
- **M. Moser, 2004, open tube model, simple analytical analysis**
- **E. Jung, 2007, lumped model for loop**
- **A. I. Hikerson, 2005, Impedance pump**
- **J. T. Ottesen, 2003, 1D model, closed loop**
- **other models to try to understand the physics of valveless pumping**

Energy-Based Lumped Model(ELM)



Symbol	meaning
n	the number of compartments in elastic tube
C_L	the left tank compartment
C_R	the right tank compartment
C_i	$i-$th compartment of elastic tube from the left tank
l_i	the length of $i-$th compartment
C^{pump}	the external pumping compartment
ρ	the fluid density
ν	the fluid viscosity
g	the gravitational constant
E	Young's modulus
A_L	the cross section area of the fluid in C_L
A_R	the cross section area of the fluid in C_R
$P_i(t)$	the representative pressure of fluid in C_i
$Q_i(t)$	the flux between fluids in C_{i-1} and C_i
$r_i(t)$	the radius of fluid in C_i
$h_i(t)$	the thickness of elastic tube wall in C_i
$H_L(t)$	the fluid height in C_L
$H_R(t)$	the fluid height in C_R
Δt	the time step for RUNGE-KUTTA method

ELM: (1) Flux model in elastic tube



energy principle

$$\dot{e}_i = \underbrace{-(P_i \bar{A}_i)v_i + (P_{i-1} \bar{A}_i)v_i}_{①} + \underbrace{\mathcal{F}\, v_i}_{②}$$

① the work done by the **pressure** force

② the work done by the **friction** force

dual lump of fluid

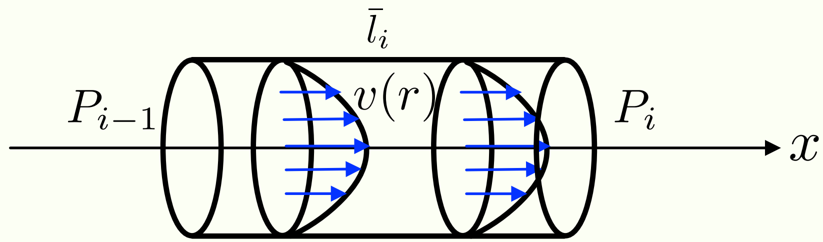
$$\bar{V}_i = \bar{A}_i \, \bar{l}_i, \quad \bar{l}_i \equiv \frac{l_{i-1} + l_i}{2}$$

$$Q_i = v_i \, \bar{A}_i$$

the energy of the dual lump of fluid

$$e_i = \frac{1}{2}\rho \, \bar{V}_i \, v_i^2 = \frac{1}{2}\rho \, Q_i^2 \, \frac{\bar{l}_i^2}{\bar{V}_i}$$

$\Delta P_i \equiv P_i - P_{i-1}$ **the pressure difference**



the velocity and pressure of the fluid in the circular tube(radius a > 0) with a pressure difference

$$v(r) = -\frac{\Delta P_i}{4\mu\,\bar{l}_i}(a^2 - r^2)$$

$$P(x) = \frac{\Delta P_i}{\bar{l}_i}\,x + C$$

the fluid flux passing across the circular tube

$$Q_i = \int_0^a 2\pi r\, v(r)\, dr = -\frac{\Delta P_i\, \bar{A}_i^2}{8\pi\,\mu\,\bar{l}_i}$$

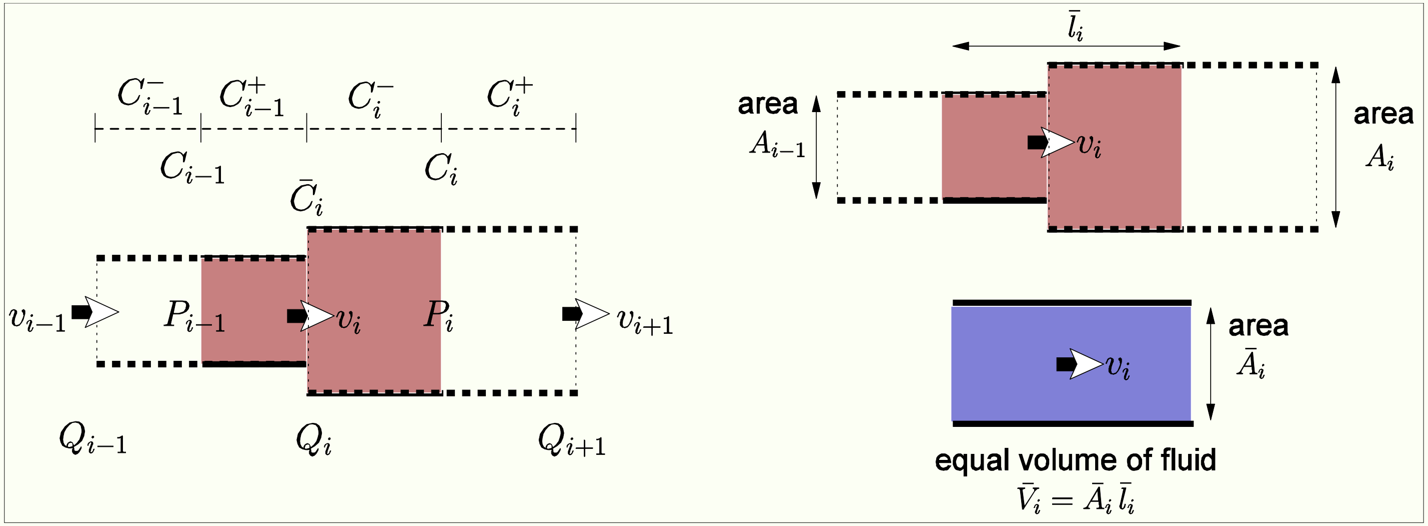
$$\Delta P_i = -\frac{8\pi\,\mu\,\bar{l}_i\, Q_i}{\bar{A}_i^2}$$

$$\mathcal{F} = \mu\,\underline{\frac{dv(r)}{dr}}\,(2\pi a\,\bar{l}_i) = \bar{A}_i\,\Delta P_i$$

the shear stress exerted on the inside of the tube wall

$$\mathcal{F} = -\frac{8\pi\,\mu\,\bar{l}_i\, Q_i}{\bar{A}_i}$$

$$\boxed{\mathcal{F}\, v_i = -\frac{8\pi\,\mu\,\bar{l}_i\, Q_i^2}{\bar{A}_i^2}}$$



$$\dot{e}_i = -(P_i - P_{i-1})\, Q_i - \frac{8\pi\, \mu\, \bar{l}_i\, Q_i^2}{\bar{A}_i^2}$$

$$e_i = \frac{1}{2}\rho\, \bar{V}_i\, v_i^2 = \frac{1}{2}\rho\, Q_i^2\, \frac{\bar{l}_i^2}{\bar{V}_i} \quad \longrightarrow \quad \dot{e}_i = \frac{1}{2}\rho\, \bar{l}_i^2 \left(2Q_i\, \dot{Q}_i\, \frac{1}{\bar{V}_i} - \bar{Q}_i^2\, \frac{\dot{\bar{V}}_i}{\bar{V}_i^2} \right)$$

$$\bar{V}_i = \frac{1}{2}\,(V_{i-1} + V_i)$$

$$\dot{V}_k = Q_k - Q_{k+1}$$

$$\dot{Q}_i = -\frac{1}{\rho \bar{\Delta}_i}(P_i - P_{i-1}) - \frac{1}{4\bar{V}_i} Q_i (Q_{i+1} - Q_{i-1}) - \frac{8\pi\nu}{\bar{A}_i} Q_i \qquad \bar{\Delta}_i \equiv \frac{\bar{V}_i}{\bar{A}_i}$$

ELM: (2) Flux model near attached fluid tanks

$$e_L = \int_0^{H_L(t)} \frac{1}{2}\,\rho\, v_L^2(t)\, A_L\, dy + \int_0^{H_L(t)} \rho\, g\, y\, A_L\, dy + \frac{1}{4}\,\rho\, V_0\, v_0^2$$

$$e_R = \int_0^{H_R(t)} \frac{1}{2}\,\rho\, v_R^2(t)\, A_R\, dy + \int_0^{H_R(t)} \rho\, g\, y\, A_R\, dy + \frac{1}{4}\,\rho\, V_{n-1}\, v_n^2$$

the kinetic energy of the fluid in tank

Potential energy

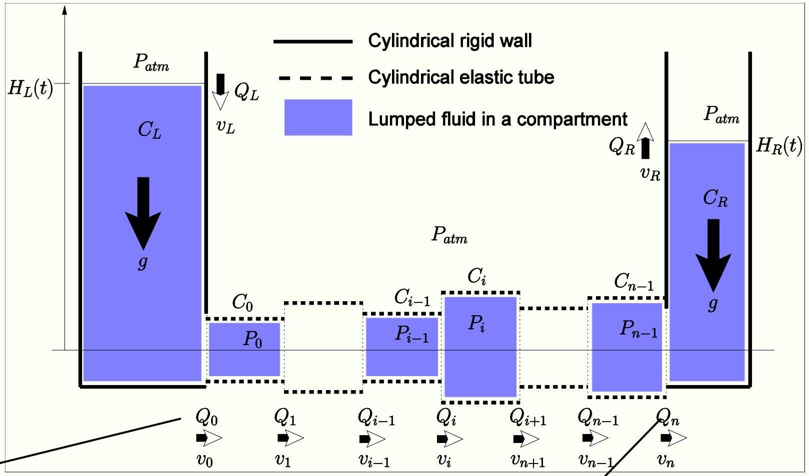
the kinetic energy of the half elastic tube attached

Using the energy-power principle,

$$\dot{e}_L = -(P_0\, A_0)\, v_0 + (P^{atm}\, A_L)\, v_L - 8\,\pi\mu\, Q_0^2\, \frac{l_0}{2\, A_0^2}$$
$$= (-P_0 + P^{atm})\, Q_0 - 8\,\pi\mu\, Q_0^2\, \frac{l_0}{2\, A_0^2}.$$

$$\dot{e}_R = (P_{n-1}\, A_{n-1})\, v_{n-1} - (P^{atm}\, A_R)\, v_R - 8\,\pi\mu\, Q_n^2\, \frac{l_{n-1}}{2\, A_{n-1}^2}$$
$$= (P_{n-1} - P^{atm})\, Q_n - 8\,\pi\mu\, Q_n^2\, \frac{l_{n-1}}{2\, A_{n-1}^2}.$$

the flux model between the tank and the adjacent compartment



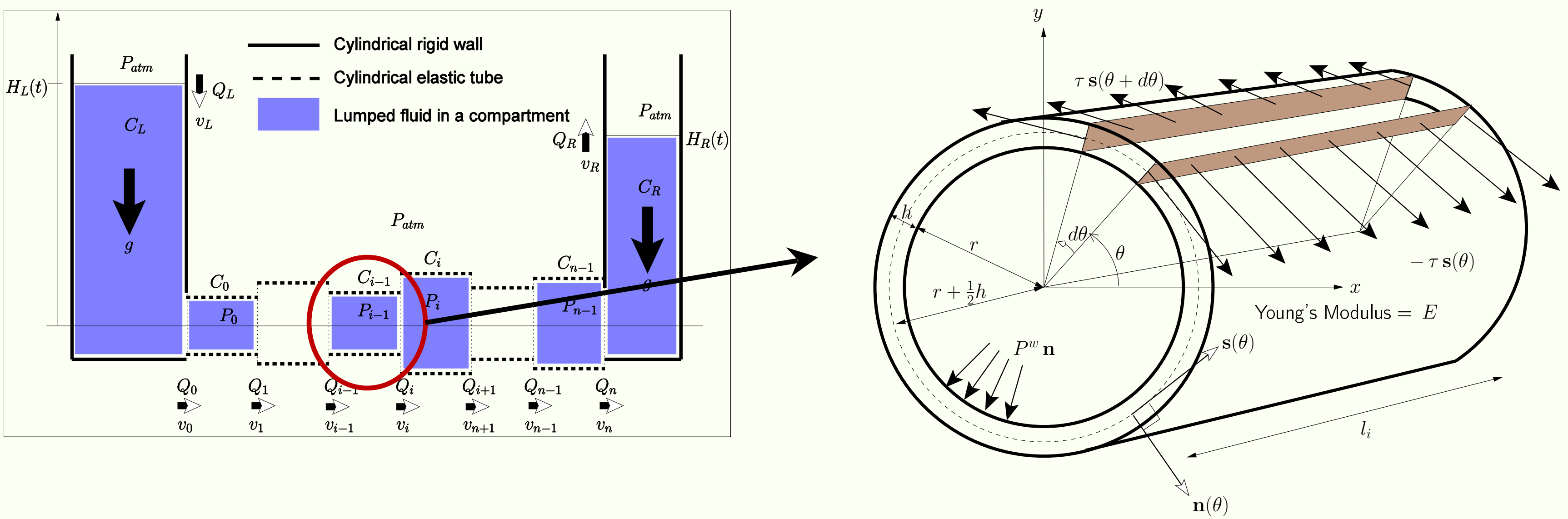
$$\dot{Q}_0 = \frac{1}{\rho\,\bar{\Delta}_L}\left[P^{atm} - P_0(t) + \rho\, g\, H_L(t)\right] + \frac{1}{4}\frac{Q_0}{\bar{\Delta}_L}\left[\frac{2\,Q_0}{A_L^2} + \frac{Q_0 - Q_1}{A_0^2}\right] - \frac{4\,\pi\,\nu}{\rho\,\bar{\Delta}_L}\,Q_0\,\frac{l_0}{A_0^2}$$

$$\bar{\Delta}_L(t) = \frac{V_L(t)}{A_L^2} + \frac{V_0}{2\,A_0^2(t)}$$

$$\dot{Q}_n = \frac{1}{\rho\,\bar{\Delta}_R}\left[-P^{atm} + P_{n-1}(t) - \rho\, g\, H_R(t)\right] + \frac{1}{4}\frac{Q_n}{\bar{\Delta}_R}\left[-\frac{2\,Q_n}{A_R^2} + \frac{Q_{n-1} - Q_n}{A_{n-1}^2}\right] - \frac{4\,\pi\,\nu}{\rho\,\bar{\Delta}_R}\,Q_n\,\frac{l_{n-1}}{A_{n-1}^2}$$

$$\bar{\Delta}_R(t) = \frac{V_R(t)}{A_R^2} + \frac{V_{n-1}}{2\,A_{n-1}^2(t)}$$

ELM: (3) Pressure model



$r(t)$ **the inner radius of cylindrical compartment**

$h(t)$ **the thickness of cylindrical tube wall**

Assumptions in the EPLM

AS1 the elastic tube in C_i preserves its cylindrical shape,

AS2 there is no change in the length l_i of C_i,

AS3 the volume of the elastic tube in C_i remains constant during deformation,

AS4 the mass of elastic tube in C_i is negligible.

ELM: (3) Pressure model

the force balance between the pressure from fluid and the tension from the tube wall

$$(P^w + dP^w)(r+dr)d\theta\, l_i\, \mathbf{n}(\theta) + (\tau + d\tau)(\mathbf{s}(\theta + d\theta) - \mathbf{s}(\theta))(h+dh)l_i = 0$$

$$\mathbf{s}(\theta + d\theta) - \mathbf{s}(\theta) = -\,\mathbf{n}\, d\theta$$

$$\rightarrow (P^w dr + r dP^w) - (\tau dh + h d\tau) = 0 \rightarrow \boxed{d(P^w r - \tau h) = 0}$$

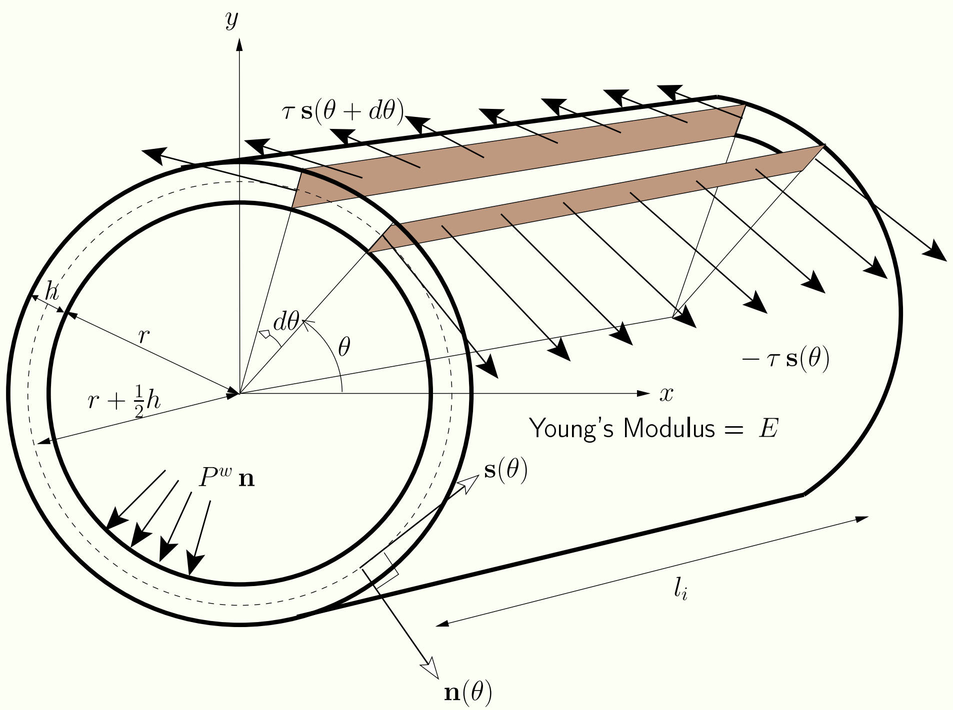
AS3 $d((r+h)^2 - r^2) = 0 \rightarrow h dr + (r+h)dh = 0$

Young's Modulus $d(\tau h) = E\, h\, d \log\left(r + \frac{1}{2}h\right)$

AS3 $d \log\left(r + \frac{1}{2}h\right) = \frac{dr}{r+h}$

$$\boxed{\dot{P}_i^w = \frac{1}{2}\left(\frac{E\, h_i}{r_i + h_i} - P_i^w\right)\frac{Q_i - Q_{i+1}}{V_i}}$$

ELM: Pressure model



AS4 $$P_i^w(t) = P_i(t) - P^{atm} - P_i^{ext}(t)$$

Elastic wall pressure

Fluid pressure

Atmospheric pressure

External pressure

System of the ELM equations

$$P_i^w(t),\ Q_i(t),\ r_i(t),\ H_L(t),\ \text{and}\ H_R(t)$$

$$\dot{P}_i^w = \frac{1}{2}\left(\frac{E\,h_i}{r_i+h_i} - P_i^w\right)\frac{Q_i - Q_{i+1}}{V_i},$$
$$i = 0, 1, \cdots, n-1,$$

$$P_i(t) = P_i^w(t) + P^{atm} + P_i^{ext}(t) \text{ for } i = 0, 1, \cdots, n-1$$

$$\dot{Q}_0 = \frac{1}{\rho\,\bar{\Delta}_L}\left[-P_0^w(t) - P_i^{ext}(t) + \rho\,g\,H_L(t)\right] + \frac{1}{2}Q_0\left[\frac{Q_0}{V_L}\frac{\Delta_L}{\bar{\Delta}_L} + \frac{Q_0 - Q_1}{V_0}\frac{\Delta_0}{2\,\bar{\Delta}_L}\right] - \frac{8\,\pi\nu}{\rho}\frac{Q_0}{A_0}\frac{\Delta_0}{2\,\bar{\Delta}_L},$$

$$\dot{Q}_i = \frac{1}{\rho\,\bar{\Delta}_i}\left[P_{i-1}^w(t) + P_{i-1}^{ext}(t) - P_i^w(t) - P_i^{ext}(t)\right] + \frac{1}{2}Q_i\,\frac{Q_{i-1} - Q_{i+1}}{2\,\bar{V}_i} - \frac{8\,\pi\,\nu}{\rho}\frac{Q_i}{\bar{A}_i},$$
$$i = 1, 2, \cdots, n-1,$$

$$\dot{Q}_n = \frac{1}{\rho\,\bar{\Delta}_R}\left[P_{n-1}^w(t) + P_{n-1}^{ext}(t) - \rho\,g\,H_R(t)\right] + \frac{1}{2}Q_n\left[-\frac{Q_n}{V_R}\frac{\Delta_R}{\bar{\Delta}_R} + \frac{Q_{n-1} - Q_n}{V_{n-1}}\frac{\Delta_{n-1}}{2\,\bar{\Delta}_R}\right] - \frac{8\,\pi\nu}{\rho}\frac{Q_n}{A_{n-1}}\frac{\Delta_{n-1}}{2\,\bar{\Delta}_R},$$

Continuity equations of EPLM

$$\dot{H}_L = -\frac{Q_0}{A_L}, \quad \dot{H}_R = \frac{Q_n}{A_R},$$

$$\dot{r}_i = \frac{r_i}{2}\frac{Q_i - Q_{i+1}}{V_i}, \text{ for } i = 0, 1, \cdots, n-1,$$

Reliability of the ELM

Physical phenomena

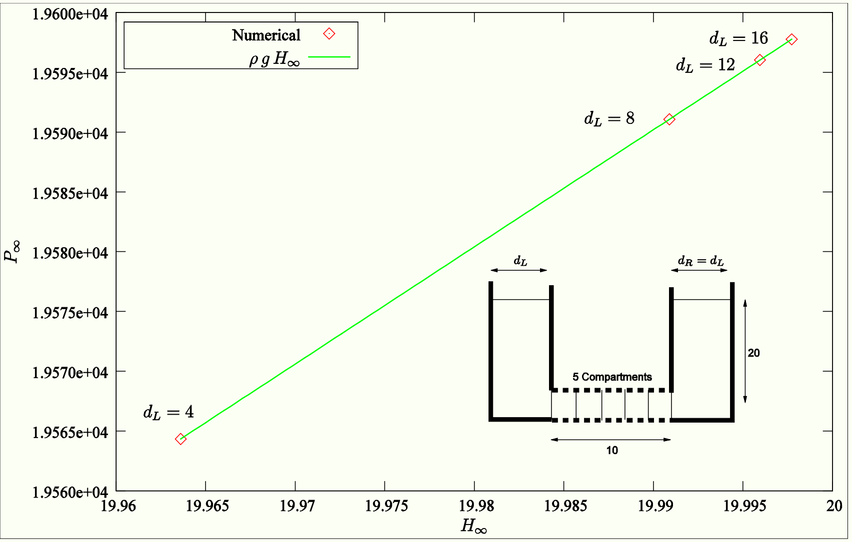
(Case i) If we consider the case where the heights of fluid in both tanks are equal at initial time and constant radius along the elastic tube is initially assumed as if there is no fluid inside, *then it is anticipated that the fluid heights in tanks, eventually move* down a little to a certain height *as t goes to infinity. Due to the total* volume conservation, the tube has to get a little fatter compared to the initial shape to make up for the loss of fluid volume from two tanks.

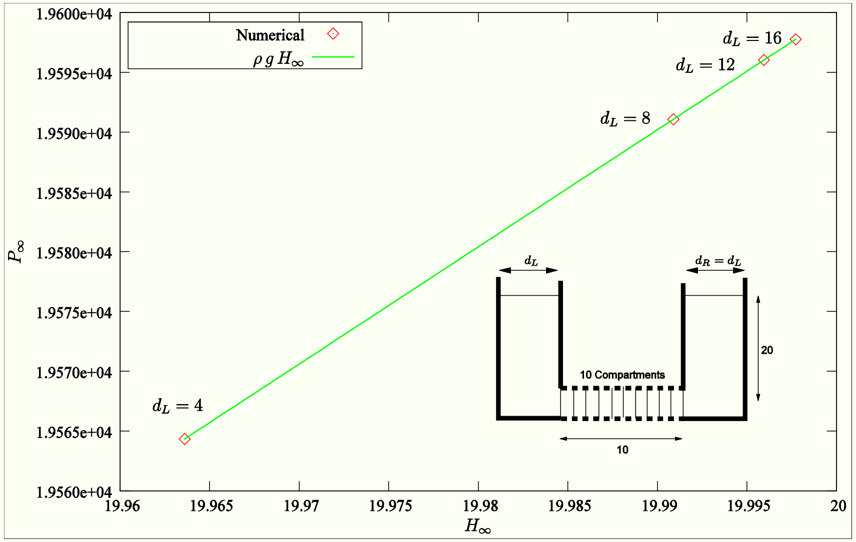
(Case ii) As an another example, it is also interesting that initially different heights of two tanks approaches an equilibrium height in the end.

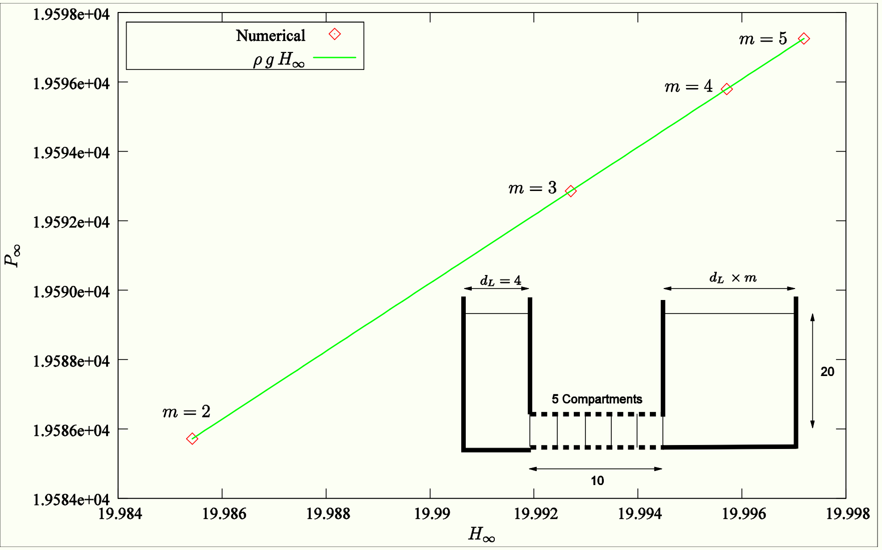
(Case iil) The EPLM has a symmetry about the change of roles of tanks.

Can our ELM produce these phenomena ? (Yes)

Reliability study: (Case i)



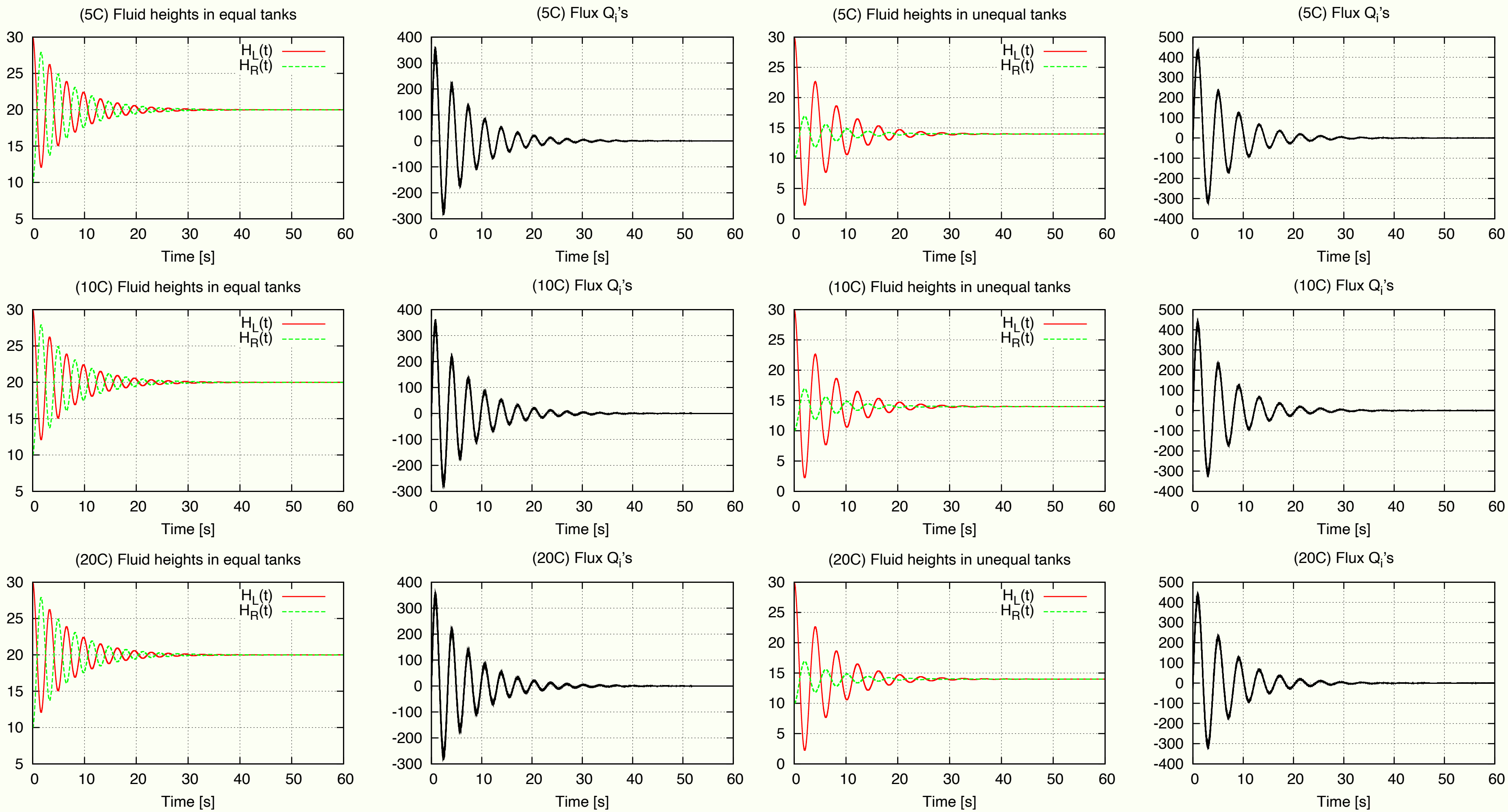




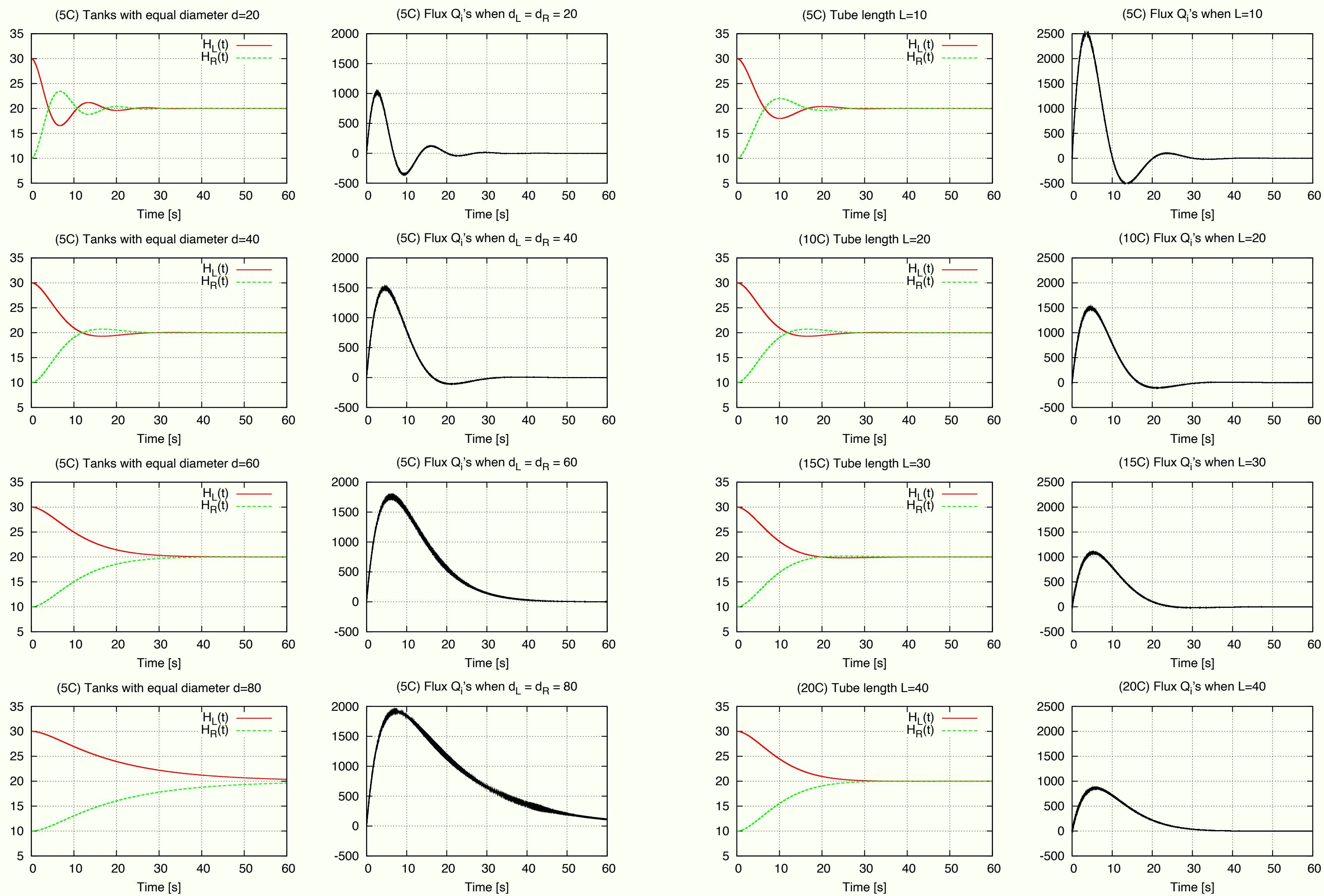
$$P_\infty \equiv \lim_{t \to \infty} P_i(t) = P^{atm} + \rho\, g\, H_\infty$$

Static pressure is recovered !!!

***Reliability study:* (Case ii)**

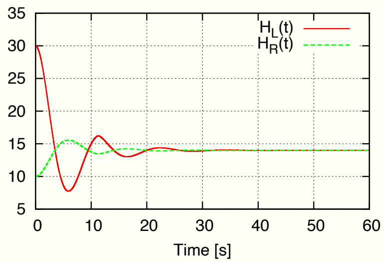


Reliability study: **(Case iii)**

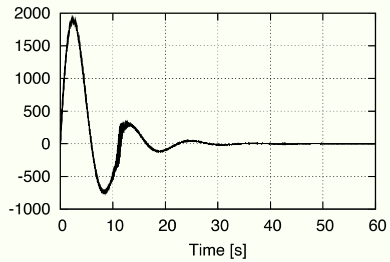


***Reliability study:* (More)**

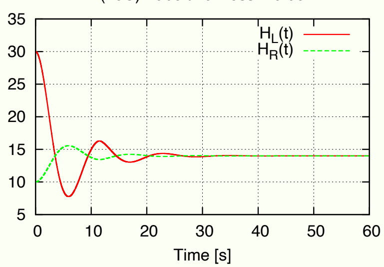
(10C) Tube thickness = 0.04



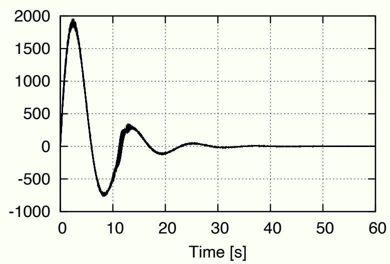
(10C) Flux Q_i's at the value 0.04



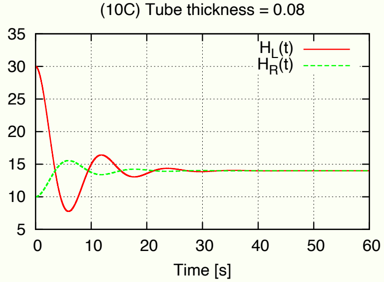
(10C) Tube thickness = 0.06



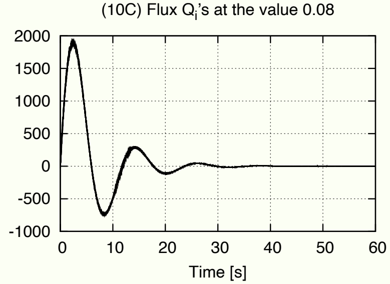
(10C) Flux Q_i's at the value 0.06



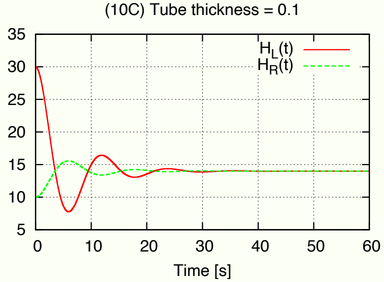
(10C) Tube thickness = 0.08



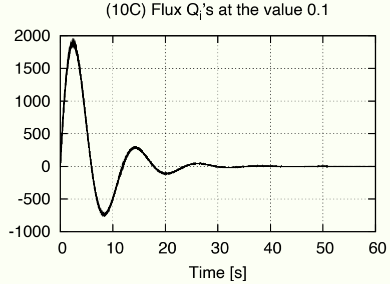
(10C) Flux Q_i's at the value 0.08



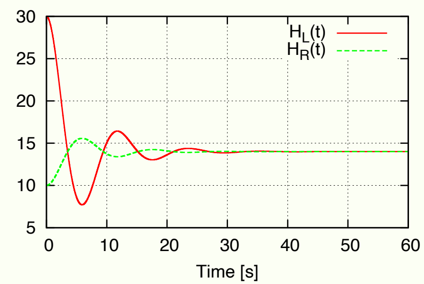
(10C) Tube thickness = 0.1



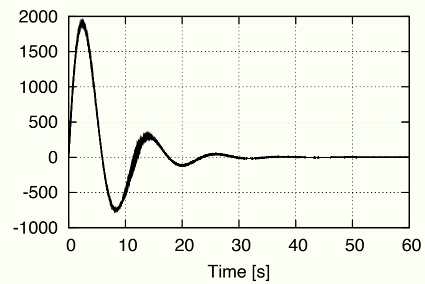
(10C) Flux Q_i's at the value 0.1



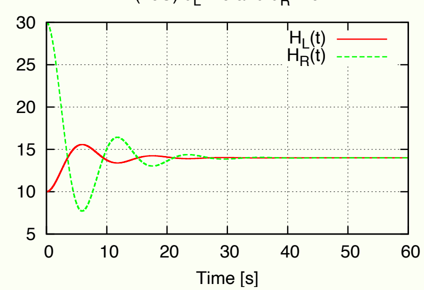
(10C) d_L=20 and d_R=40



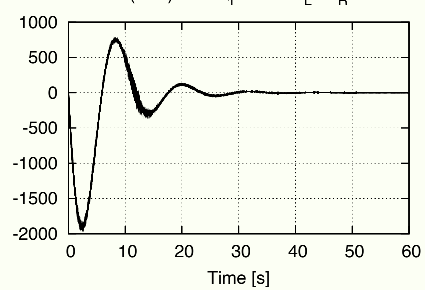
(10C) Flux Q_i's when $r_L < r_R$



(10C) d_L=40 and d_R=20



(10C) Flux Q_i's when $r_L > r_R$



Energy Conservation of the ELM

Fluid energy in the tube
Elastic energy of the tube
Total fluid energy in the left tank
Total fluid energy in the right tank

$$e^{fluid}(t) = \frac{1}{2}\,\rho\,Q_0^2(t)\,\bar{\Delta}_0 + \sum_{i=1}^{n-1}\frac{1}{2}\,\rho\,Q_i^2(t)\,\bar{\Delta}_i(t) + \frac{1}{2}\,\rho\,Q_n^2(t)\,\bar{\Delta}_n$$

$$e^{elastic}(t) = \sum_{i=0}^{n-1}\int_{r^*}^{r_i(t)} P_i^w(\eta)\,(2\,\pi\,\eta\,l_i)\,d\eta$$

$$e_L^{tank}(t) = \frac{1}{2}\,\rho\,Q_0^2(t)\,\Delta_L + \frac{1}{2}\rho\,g\,A_L\,H_L^2(t)$$

$$e_R^{tank}(t) = \frac{1}{2}\,\rho\,Q_n^2(t)\,\Delta_R + \frac{1}{2}\rho\,g\,A_R\,H_R^2(t),$$

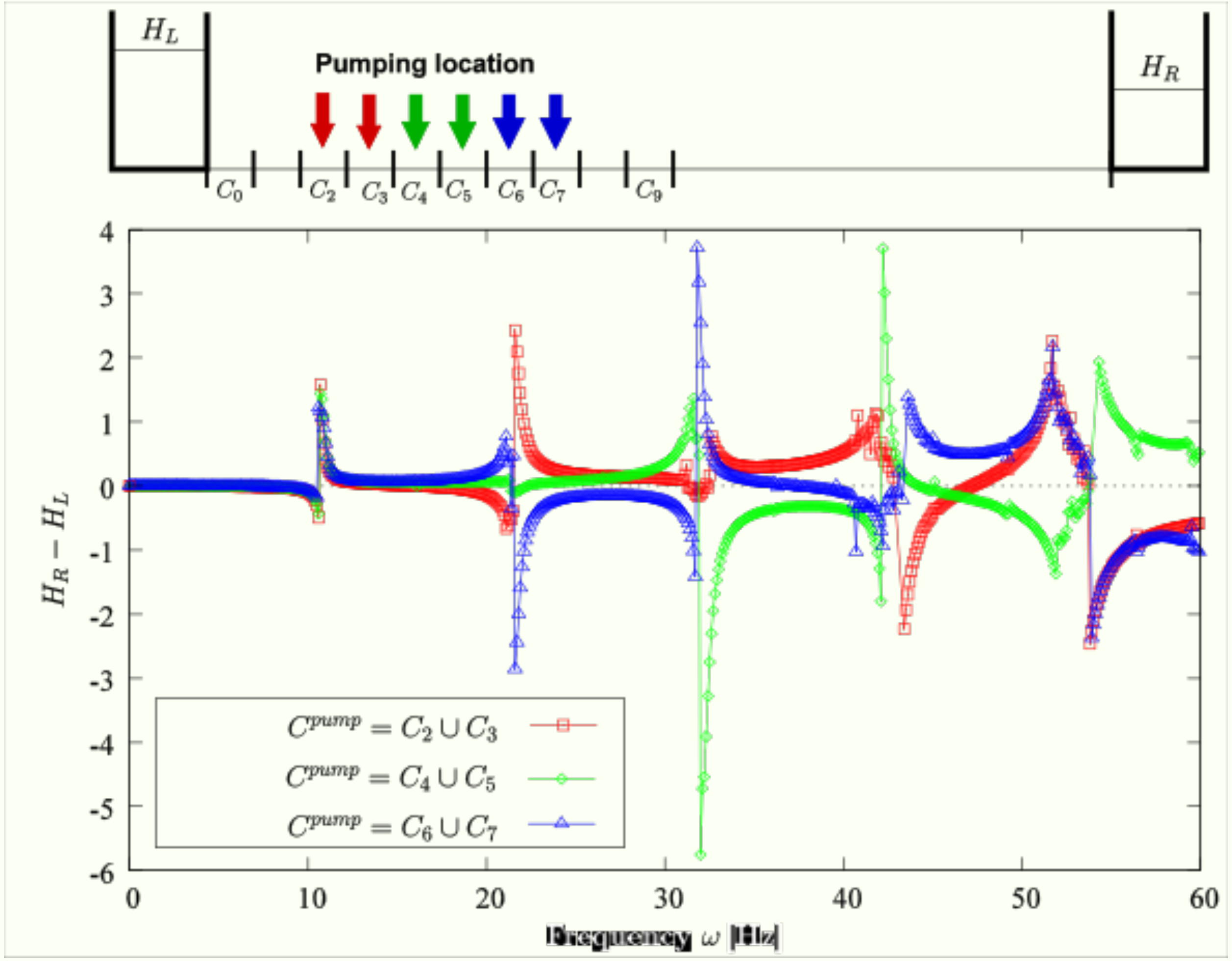
$$
\begin{aligned}
\frac{d}{dt} e^{elastic} &= \sum_{i=0}^{n-1} 2\,\pi\, r_i\, \dot{r}_i\, l_i\, P_i^w(t) = \sum_{i=0}^{n-1} P_i^w(t)\, \frac{d}{dt}(\pi\, l_i\, r_i^2(t)) \\
&= \sum_{i=0}^{n-1} (Q_i - Q_{i+1})\, (P_i(t) - P^{atm} - P_i^{ext}) \\
&= -(Q_0 - Q_n)\, P^{atm} + \sum_{i=0}^{n-1} (Q_i - Q_{i+1})\, (P_i - P_i^{ext}).
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dt}(e^{fluid} + e_L^{tank} + e_R^{tank}) \\
&\quad = (-P_0 + P^{atm})\, Q_0 + \sum_{i=1}^{n-1} (P_{i-1} - P_i)\, Q_i + (P_{n-1} - P^{atm})\, Q_n \\
&\qquad - \left[8\,\pi\,\nu\, Q_0^2(t)\, \frac{\bar{\Delta}_0}{A_0} + \sum_{i=1}^{n-1} 8\,\pi\,\nu\, Q_i^2(t)\, \frac{\bar{\Delta}_i}{\bar{A}_i} + 8\,\pi\,\nu\, Q_n^2(t)\, \frac{\bar{\Delta}_n}{A_{n-1}} \right] \\
&\quad = (Q_0 - Q_n)\, P^{atm} - \sum_{i=0}^{n-1} (Q_i - Q_{i+1})\, P_i - \sum_{i=0}^{n} 8\,\pi\,\nu\, Q_i^2(t)\, \frac{\bar{\Delta}_i}{\bar{A}_i},
\end{aligned}
$$

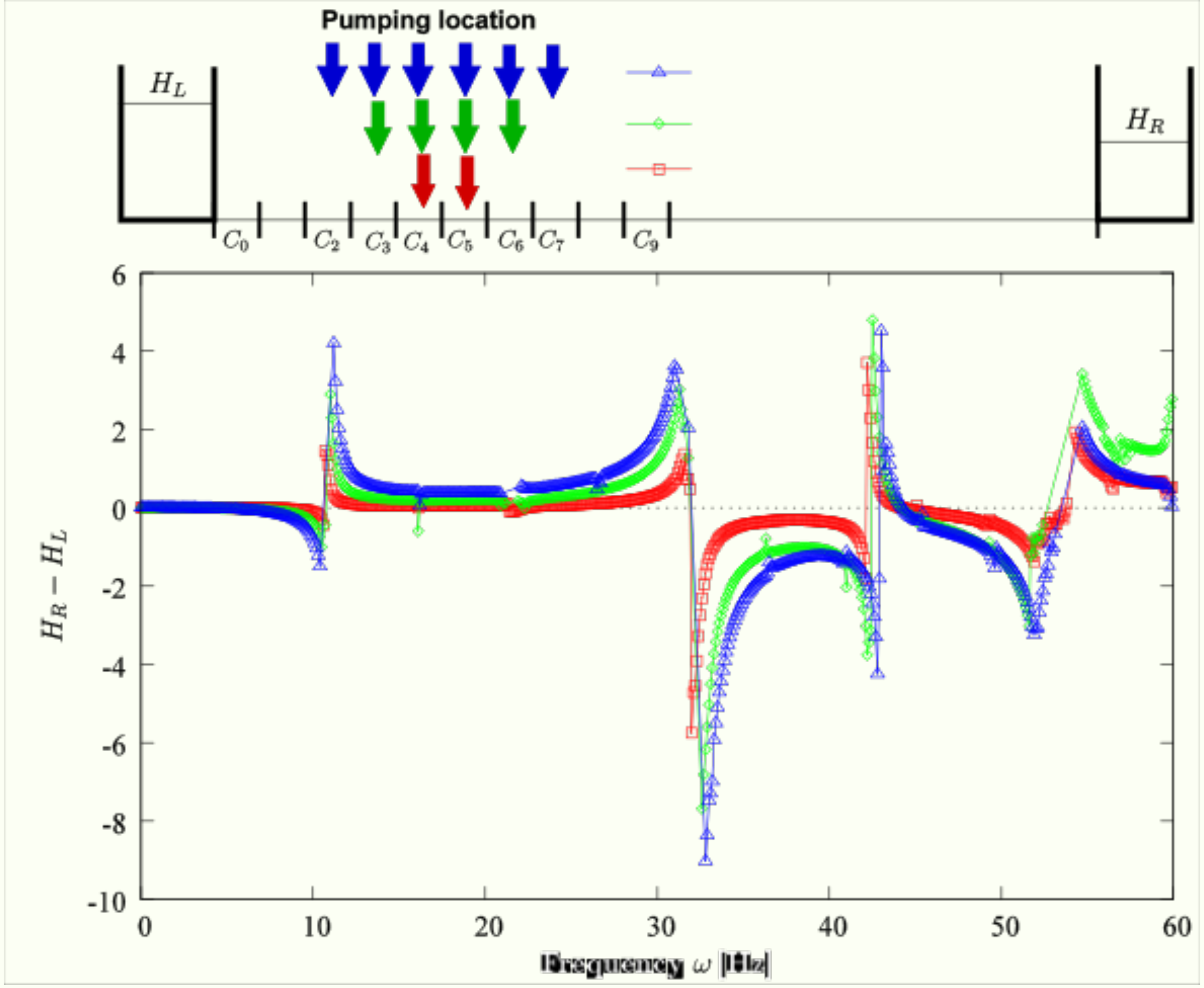
$$e^{total} \equiv e^{fluid} + e^{elastic} + e_L^{tank} + e_R^{tank}$$

$$\dot{e}^{total}(t) = \underbrace{-\sum_{i=0}^{n-1}(Q_i - Q_{i+1})\, P_i^{ext}(t)}_{\text{Pumping power}} \underbrace{-\sum_{i=0}^{n} 8\,\pi\,\nu\, Q_i^2(t)\, \frac{\bar{\Delta}_i}{\bar{A}_i}}_{\text{Friction power}}$$

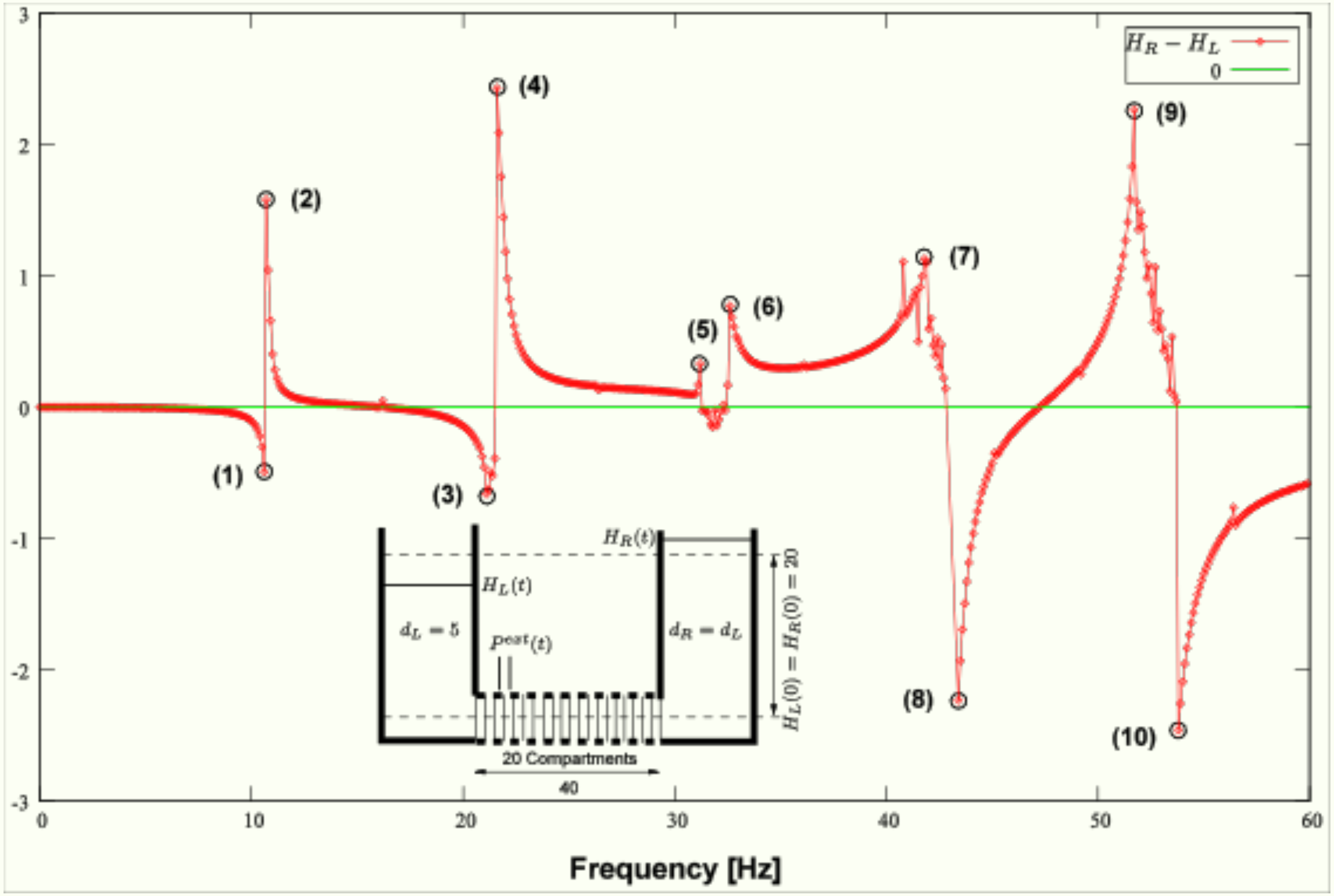
Occurrence of valveless pumping using the ELM



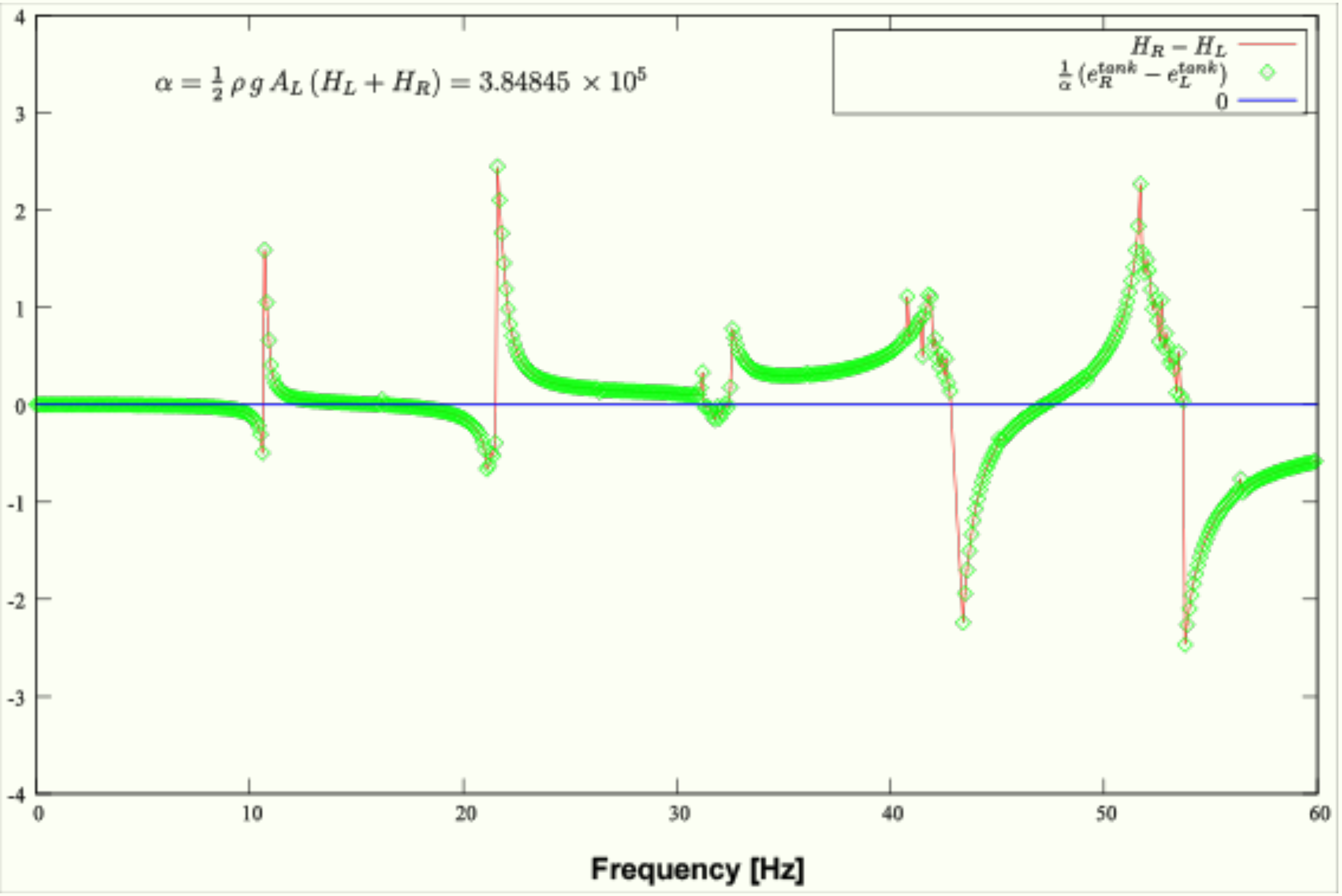
Occurrence of valveless pumping using the EPLM



Occurrence of valveless pumping using the ELM



Energy Storing Effect by valveless pumping

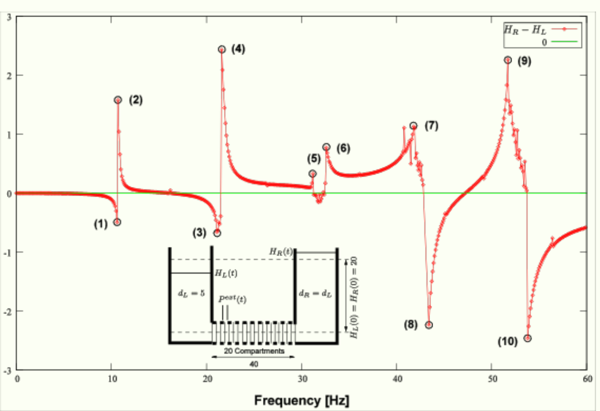


Index for the occurrence of pumping effect proposed by the ELM

$$(\Delta P)_k \equiv \int_{C_k} P_x \, dx = \bar{P}_k - \bar{P}_{k-1},$$

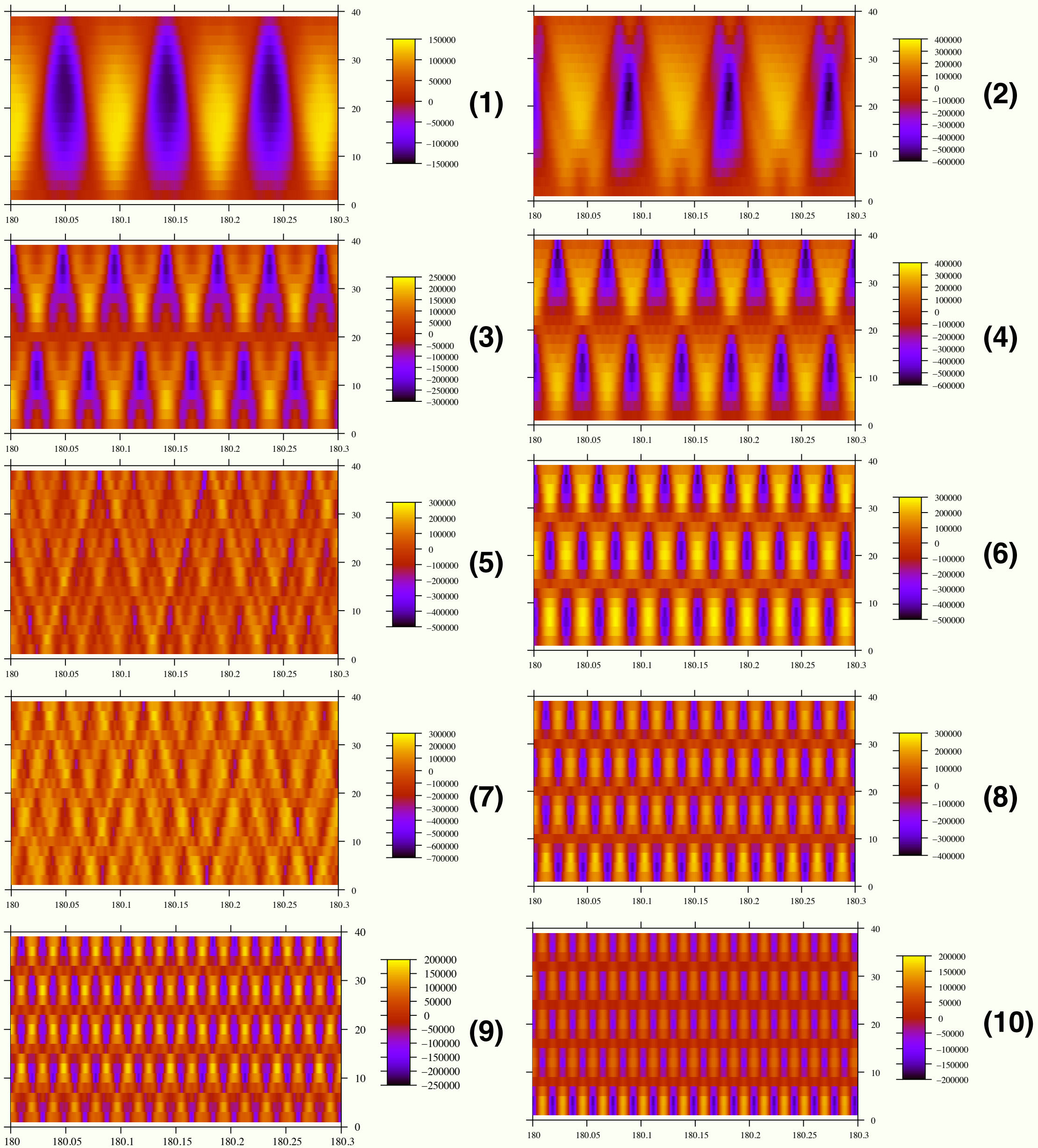
$$\bar{P}_k \, \bar{V}_k = P_{k-1} \left(\frac{1}{2} V_{i-1} \right) + P_k \left(\frac{1}{2} V_k \right)$$

$$D^{power} \equiv \overline{\sum_{k \in \Lambda^{ext}} (\Delta P)_k(t) \, P_k^{ext}(t)}.$$

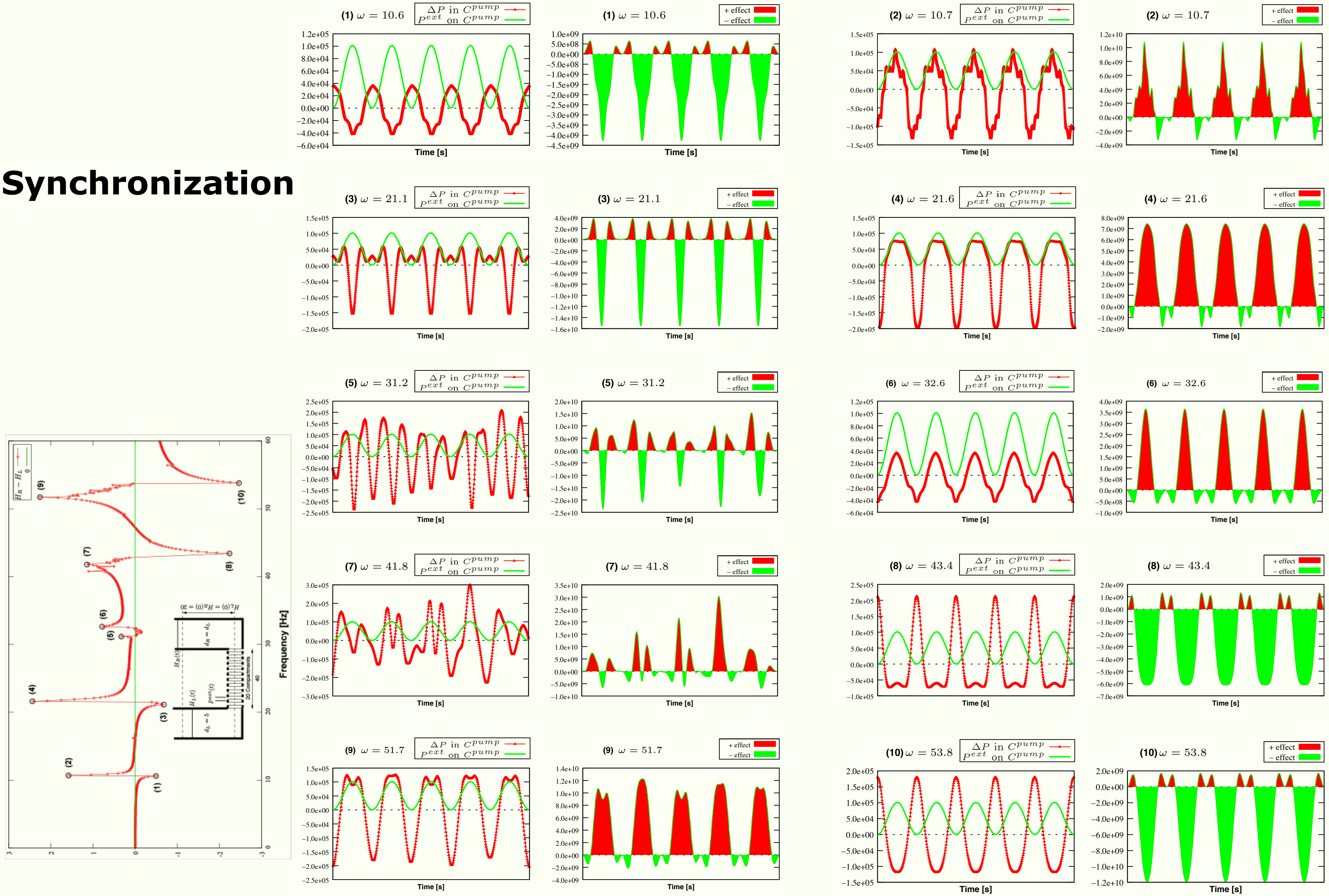


peak number	external pumping frequency (ω)	index D^{power} [$\times 10^9$]	$H_R - H_L$
(1)	10.6	−1.0143	−0.4947
(2)	10.7	+2.1062	+1.5794
(3)	21.1	−1.4345	−0.6632
(4)	21.6	+2.7677	+2.4324
(5)	31.2	−0.0579 (transient)	+0.2842
(6)	32.6	+0.7643	+0.7718
(7)	41.8	+2.7812 (transient)	+1.0849
(8)	43.4	−2.7266	−2.2331
(9)	51.7	+4.0763	+2.2615
(10)	53.8	−3.6763	−2.4610

Fluid pressure inside the elastic tube



Synchronization



Conclusions

(1) Energy-Based Lumped Model(ELM) is developed.
(2) Energy conservation is proved in ELM.
(3) It passes the reliability tests.
(4) The pumping effect can be understood through the ELM.

Thank you very much!